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DEFORMATION OF AN ELLIPSOIDAL FERROGEL SAMPLE IN A UNIFORM MAGNETIC FIELD

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The elongation of a ferroelastic material sample (whose initial shape is a sphere or an ellipsoid of revolution) under the action of an external magnetic field is studied in an in approximation of small strains. For a sphere, there is a classical estimate obtained under the assumption that elongating in the direction of the field, it becomes a spheroid and the stress and strain fields remain uniform. In the present calculation, it is assumed that the body is an ellipsoid (a sphere in a particular case) only in the absence of an external field; the shape of the sample in the presence of a field is not specified in advance but is found from the condition of balance of surface forces (elastic and magnetic). For the spherical case, the problem is solved exactly: it is shown, that the contour of the deformed body is described by a third-order algebraic equation. The case where the initial configuration is an ellipsoid of revolution is studied numerically. It is shown that in all versions, the refined solution leads to an appreciable increase in the elongation of the sample compared to the classical estimate.

Key words: *small strains, ferrogel, magnetoelasticity, magnetic-deformation effect.*

Introduction. The term ferroelast or magnetic elastomer refers to a composite system consisting of an elastic (viscoelastic) polymer matrix filled with a fine ferromagnetic material. Recently, there has been considerable interest in soft ferroelasts (ferrogels) [1–6] whose Young's moduli $\lesssim 10^4$ Pa. They are candidates for use as materials for artificial muscles, active elements of micromanipulators, adaptive coatings, etc. All applications of ferrogels are based on the magnetic-deformation effect (MDE), whose essence is that a sample placed in a magnetic field changes the initial shape and the strain can reach many tens of percent [7, 8].

The simplest version of the solution of the MDE problem is readily obtained if one takes the solution from [9] for the deformation problem for a spherical sample of radius R subjected to an external electric field E_0 and then rewrite it using the analogy between the magnetostatic and electrostatic potentials. As a result, we have

$$\frac{a_l - b_l}{R} = \frac{\varepsilon_0}{10G}\Big(\frac{æE_0}{1 + æ/3}\Big)^2 = \frac{P^2}{10G\varepsilon_0}, \tag{1}$$

where a_l and b_l are the semiaxes of the cross section of the ellipsoid obtained as a result of deformation, P is the polarization inside the sphere, G is the shear modulus, $æ$ is the dielectric susceptibility of the material, and ε_0 is the dielectric constant. As follows from formula (1), the dielectric striction effect is even in the field.

Below, the elongation is characterized by the parameter ε, which is equal to the relative change in the distance between the geometrical poles of the sample in the direction of the applied field. In this representation, the striction effect (1) is expressed by the relation

$$\varepsilon = a_l/R - 1 = P^2/(15G\varepsilon_0), \tag{2}$$

which takes into account that at small strains of the incompressible sphere, $(a_l - b_l)/R \simeq 3(a_l - R)/(2R)$. The transition from relations (1) or (2) to the magnetic case, which will be called the magnetic-deformation effect is obvious: it suffices to replace the electric-field stress and polarization vectors by the corresponding characteristics

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of the magnetic field and the dielectric susceptibility æ by the magnetic susceptibility χ. Subsequently, we assume that this transition is implemented and use relation (2), appropriately redesignated, as the classical estimate of the magnetic-deformation effect in a uniform field.

General Formulation of the Problem of Magnetic-Deformation Effect. The assumption that a magnetoelastic spheroid always remains such (a figure of the second order according to the classification adopted in analytical geometry), which underlies the classical solution, is extremely convenient from a mathematical point of view. However, in [9], the physical admissibility of this postulate is not discussed. At the same time, in the solution of the problem in the full formulation, the shape taken by the body upon magnetization should be found without additional conditions. From this it follows that the postulate of ellipsoidality is likely to be incompatible with the exact solution. To elucidate this question, we write the complete system of MDE equations, relating in it the magnetostatic and elastic problems. We begin with the magnetostatic part. In the absence of currents, the magnetic-field strength vector $\boldsymbol{H}$ can be represented as the gradient of the scalar function ψ: $\boldsymbol{H} = \boldsymbol{H}_0 - \nabla\psi$ ($\boldsymbol{H}_0$ is the applied external field). From the solenoidality condition $\nabla \cdot (\boldsymbol{H} + \boldsymbol{M}) = 0$, we have

$$\Delta\psi = \nabla \cdot \boldsymbol{M}, \tag{3}$$

where $\boldsymbol{M}$ is the magnetization vector.

The relationship between the magnetic vectors on the sample surface follows from the continuity conditions for the normal induction component and the tangential component of the magnetic-field strength:

$$\frac{\partial\psi^{(i)}}{\partial n}\Big|_\Gamma - \frac{\partial\psi^{(e)}}{\partial n}\Big|_\Gamma = \boldsymbol{M}\cdot\boldsymbol{n}\Big|_\Gamma, \qquad \psi^{(i)}\Big|_\Gamma = \psi^{(e)}\Big|_\Gamma. \tag{4}$$

Here $\boldsymbol{n}$ is the outward normal vector and Γ is the boundary of the sample. Here and below, the superscripts (i) and (e) denote the values of the quantities inside the sample and in the external region, respectively.

The balance condition for the elastic forces inside the sample is expressed by the equation

$$\nabla \cdot T + \mu_0(\boldsymbol{M}\cdot\nabla)\boldsymbol{H} = 0, \tag{5}$$

where T is the Cauchy stress tensor and $\mu_0(\boldsymbol{M}\cdot\nabla)\boldsymbol{H}$ is the ponderomotive force due to the nonuniformity of the magnetic field inside the sample. In equilibrium, the following condition should be satisfied on the boundary of the sample:

$$\boldsymbol{n}\cdot T\Big|_\Gamma = (1/2)\mu_0 M_n^2 \boldsymbol{n}\Big|_\Gamma. \tag{6}$$

It expresses the balance of the external and internal pressures; $M_n = \boldsymbol{M}\cdot\boldsymbol{n}$ is the normal component of the magnetization vector.

Relations (3)–(6) together with the equations of state $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{H}, e)$ and $T = T(\boldsymbol{H}, e)$ (e is the strain tensor) form the closed system of equations of the static magnetoelastic problem, whose solution should completely define the shape taken by the ferroelastic sample in the specified external field and describe the magnetic-field distribution and the stress–strain state inside the ferroelastic body.

Magnetic-Deformation Effect at Small Strains of the Ellipsoid. In the initial state (with no external field), let the ferroelastic sample have the shape of an ellipsoid of revolution. We place the origin of the coordinate system at the center of the sample. The external uniform field $\boldsymbol{H}_0$ is directed along the principal axis of symmetry of the sample, which is chosen to be the Oz axis of cylindrical coordinates; the basis vectors of the latter are denoted by $\boldsymbol{\epsilon}_\rho$, $\boldsymbol{\epsilon}_\theta$, and $\boldsymbol{\epsilon}_z$. As the dimensionless characteristics of the ellipsoid, we use the ratio a/b of the lengths of its semiaxes, where a is the principal semiaxis.

We consider the initial stage of the magnetic-deformation effect, i.e., we assume that the strains are small. In this case, the magnetostatic force does not depend on the elongation and can be found under the assumption that the sample shape is unperturbed. Because of this, the problems of calculating the internal magnetic field and the mechanical-stress and strain fields are split and can be solved sequentially.

The magnetostatic problem for an ellipsoid has the well-known analytical solution, according to which the field inside the sample is uniform, is directed along the Oz axis, and is equal to

$$H = H_0 - M(H)N(x). \tag{7}$$

Here N is the demagnetizing factor along the Oz axis and x is the eccentricity of the meridian section of the body. For a prolate spheroid ($a > b$), we have

$$N(x) = (1 - x^2)(\operatorname{arctanh} x - x)x^{-3}, \qquad x = \sqrt{1 - b^2/a^2};$$

for an oblate spheroid ($a < b$), the demagnetizing factor is written as

$$N(x) = (1 + x^2)(x - \operatorname{arctanh} x)x^{-3}, \qquad x = \sqrt{b^2/a^2 - 1}.$$

For the magnetization of the ferroelastic sample, we use the Langevin law

$$\boldsymbol{M}(\boldsymbol{H}) = M_0 L(\mu_0 mH/kT)\boldsymbol{h}, \qquad L(\xi) \equiv \coth \xi - 1/\xi, \tag{8}$$

thus assuming that the magnetic phase of the material consists of superparamagnetic one-domain or subdomain particles. In Eq. (8), the magnetic moment of a separate particle $m = Iv$ is defined by the product of its volume v by the magnetization I of the ferromagnetic material; μ_0 is the magnetic constant. The saturation magnetization of the ferroelastic sample is written as $M_0 = nm$, where n is the number concentration of magnetic particles. Physically, the argument of the Langevin function is the ratio of the orientational energy of the magnetic moment of the particle in the applied field $\boldsymbol{H} = H\boldsymbol{h}$ ($\boldsymbol{h}$ is the unit vector) to the thermal energy kT (k is Boltzmann's constant).

We convert to dimensionless variables by introducing the fields $\xi = \mu_0 mH/(kT)$ and $\xi_0 = \mu_0 mH_0/(kT)$. In the new notation, Eq. (7), which defines the internal field in the ellipsoid, in view of (8), becomes

$$\xi = \xi_0 - 3\chi_0 N(x)L(\xi), \tag{9}$$

where $\chi_0 = \mu_0 mM_0/(3kT)$ is the initial magnetic susceptibility. We note that Eq. (9) is nonlinear in ξ and has no analytical solution.

To describe the elastic behavior of the material, we use small strain theory; i.e., as the equation of state, we use Hooke's law and define the relationship between the strain tensor e and the displacement vector $\boldsymbol{u}$ by the linear relations

$$T = \lambda I_1(e)g + 2Ge, \qquad e = (\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^{\mathrm{t}})/2, \tag{10}$$

where $I_1(e)$ is the first invariant of the strain tensor, g is the unit tensor, and λ and G are Lamé's constants.

In the case of a uniform external field, the magnetic field inside the ellipsoid is also uniform, because of which mass forces are absent. Under these assumptions, Eq. (5) becomes $\nabla \cdot T = 0$. Condition (6) on the sample boundary Γ (taking into account that $M_n^2 = M^2 n_z^2$) and the boundary conditions on the ρ and z axes formulated taking into account the symmetry of the problem become

$$\boldsymbol{n} \cdot T\Big|_{\Gamma} = (1/2)\mu_0 M^2 n_z^2 \boldsymbol{n}\Big|_{\Gamma}, \qquad u_\rho\Big|_{\rho=0} = u_z\Big|_{z=0} = 0, \qquad T_{\rho z}\Big|_{z=0} = T_{\rho z}\Big|_{\rho=0} = 0. \tag{11}$$

We choose the shear modulus G as the uniform scale for the magnetization, stress tensor, and the coefficient λ; i.e., we set $\tilde{\boldsymbol{M}} = \sqrt{G/\mu_0}\,\boldsymbol{M}$, $\tilde{T} = T/G$, and $\beta = \lambda/G$. The last of the introduced dimensionless parameters characterizes the compressibility of the material: the limit $\beta \to \infty$ corresponds to the incompressible ferroelast. In the calculations below, the tilde sign is omitted; therefore, reverse conversion to dimensional variables will be specified separately.

Extending the definition (2) to the case of a sample whose initial configuration is an ellipsoid of revolution, as the characteristic of elongation, we shall use the quantity

$$\varepsilon = u_z(\rho = 0, z = a)/a, \tag{12}$$

which is equal to the relative change in the distance between the poles of the sample in the Oz direction (the field direction); here a is the semiaxis along which the applied field is directed.

Magnetic-Deformation Effect for Homogeneous Deformation of a Sphere. We use the general equations given above to describe the strain of a ferroelastic sphere; the radius of the sphere is considered unit. The demagnetization factor of the sphere equals to $1/3$ since, according to (9), the magnetic field inside the sample is defined by the equation $\xi = \xi_0 - \chi_0 L(\xi)$. We now consider the elastic problem assuming a linear distribution of displacements in the form

$$u_\rho = A\rho, \qquad u_z = Bz. \tag{13}$$

In cylindrical coordinates, the components of the displacement vector and the strain tensor are linked by the relation

$$e_{\rho z} = \frac{1}{2}\Big(\frac{\partial u_z}{\partial \rho} + \frac{\partial u_\rho}{\partial z}\Big), \qquad e_{\rho\rho} = \frac{\partial u_\rho}{\partial \rho}, \qquad e_{zz} = \frac{\partial u_z}{\partial z}, \qquad e_{\theta\theta} = \frac{u_\rho}{\rho}. \tag{14}$$

Substitution of linear relations (13) into the above relation shows that only the diagonal components of the strain tensor are nonzero and that inside the sample they are homogeneous: $e_{\rho\rho} = e_{\theta\theta} = A$ and $e_{zz} = B$. In view of this, from the constitutive relation (10), we obtain

$$T_{zz} = \beta I_1(e) + 2e_{zz}, \qquad T_{\rho z} = 2e_{\rho z}, \qquad T_{\rho\rho} = \beta I_1(e) + 2e_{\rho\rho},$$
$$T_{\theta\theta} = \beta I_1(e) + 2e_{\theta\theta}, \qquad I_1(e) = e_{\rho\rho} + e_{zz} + e_{\theta\theta}, \tag{15}$$

whence it is evident that the stress tensor T also has a diagonal form and is homogeneous inside the sample. Then, according to (12), the elongation ε becomes

$$\varepsilon = u_z(\rho = 0, z = 1), \tag{16}$$

and in the case of homogeneous deformation, it becomes $\varepsilon = Bz|_{z=1} = e_{zz}$.

For the examined magnetoelastic problem, we write the virtual work principle:

$$\frac{1}{2} M^2 \int\limits_{\Gamma} n_z^2 \boldsymbol{n} \cdot \delta \boldsymbol{u} \, dS = \int\limits_{V^{(i)}} T \cdot \delta e \, dV. \tag{17}$$

In the case of uniform stress and strain fields, integration over the volume of the sphere of unit radius reduces to multiplication by $4\pi/3$; the surface integral on the left side of Eq. (17) is easily calculated in spherical coordinates. Substituting Hooke's law (10) and the strain field (13) into Eq. (17), assuming that the variations δA and δB are independent, and setting the coefficients at them equal to zero, we find the equilibrium strain tensor

$$e_{\rho\rho} = e_{\theta\theta} = -\frac{1}{10} M^2 \frac{\beta - 1}{3\beta + 2}, \qquad e_{zz} = \frac{1}{10} M^2 \frac{2\beta + 3}{3\beta + 2}.$$

As $\beta \to \infty$, i.e., in the case of an incompressible material, this implies that the elongation parameter (in dimensional form) is equal to

$$\varepsilon = e_{zz} = \mu_0 M^2/(15G). \tag{18}$$

A comparison of this result with formula (2) proves that the above consideration and the classical estimate are identical.

Magnetic-Deformation Effect for Homogeneous Deformation of an Ellipsoid. The classical problem [9] is easy to extend to the case where the sample in the initial state ($H_0 = 0$) has the shape of an arbitrary ellipsoid of revolution. Keeping the homogeneous deformation hypothesis (13), we use Eq. (17) and choose the length of the spheroid semiaxis b as the measurement unit. The integral over the sample surface is taken in elliptic coordinates, and the integral over the volume reduces to multiplication of the integrand by the volume of the body $4\pi ab^2/3$. Requiring that the coefficients at the variations δA and δB vanish, for a prolate spheroid ($a > b$) we obtain a system of two linear equations, whence we find the components of the equilibrium strain tensor and the elongation parameter in the form

$$e_{\rho\rho} = e_{\theta\theta} = -M^2 \frac{(x^2 - 1)\{3[\beta(x^2 - 3) + 2(x^2 - 1)] \operatorname{arctanh} x + 3x(3\beta + 2) - 4x^3\}}{8(3\beta + 2)x^5},$$
$$\varepsilon = e_{zz} = M^2 \frac{(x^2 - 1)\{3[\beta(x^2 - 3) - 2] \operatorname{arctanh} x + 3x(3\beta + 2) + 2x^3\}}{4(3\beta + 2)x^5}, \tag{19}$$

where $x = \sqrt{1 - b^2/a^2}$ is the eccentricity of the meridian section. For an incompressible material ($\beta \to \infty$), from relations (19) we have

$$\varepsilon = e_{zz} = -2e_{\rho\rho} = -2e_{\theta\theta} = M^2 \frac{(x^2 - 1)[(x^2 - 3) \operatorname{arctanh} x + 3x]}{4x^5}, \tag{20}$$

which at the limit $x \to 0$, i.e., for a spherical sample becomes (18); in the comparison, it is necessary to take into account that formula (20) is written in dimensionless units.

Requiring that the variations vanish, for an oblate spheroid ($a < b$) we obtain

$$e_{\rho\rho} = e_{\theta\theta} = -M^2 \, \frac{(1+x^2)\{3[\beta(x^2+3) + 2(x^2+1)] \operatorname{arctanh} x - 3x(3\beta+2) - 4x^3\}}{8(2+3\beta)x^5},$$
$$\varepsilon = e_{zz} = M^2 \, \frac{(1+x^2)\{3[\beta(x^2+3)+2] \operatorname{arctanh} x - 3x(3\beta+2) + 2x^3\}}{4(3\beta+2)x^5}, \tag{21}$$

where $x = \sqrt{b^2/a^2 - 1}$. For an incompressible material ($\beta \to \infty$), from the latter expression, we have

$$\varepsilon = e_{zz} = -2e_{\rho\rho} = -2e_{\theta\theta} = M^2 \, \frac{(x^2+1)[(x^2+3)\operatorname{arctanh} x - 3x]}{4x^5}. \tag{22}$$

In the limiting cases of needlelike ($b/a \to 0$) and disklike ($a/b \to 0$) bodies, using suitable expansions in formulas (22) and (20), we obtain

$$\varepsilon = \frac{1}{4}\, M^2 \begin{cases} (b^2/a^2) \ln [4a^2/(e^3 b^2)] + O(b^4/a^4) & \text{at} \quad b/a \to 0, \\ a/(2b) + O(a^2/b^2) & \text{at} \quad a/b \to 0. \end{cases} \tag{23}$$

It is obvious that in both cases, the magnetic-deformation effect in the ellipsoid is vanishingly small irrespective of the particular magnetization law.

Exact Solution of the Linear Problem of the Magnetic-Deformation Effect for a Sphere. We consider the general situation, in which, unlike in the classical case, the requirement of stress and strain homogeneity is not formulated. In other words, the body configuration resulting from the MDE is not postulated but is found from the solution of the elastic problem. We represent the displacement $\boldsymbol{u}(\rho, z)$ at an arbitrary point of the sample in the form of a series in the powers of the coordinates ρ and z and impose the following symmetry conditions on the desired solution: u_ρ depends only on the odd powers of ρ and even powers of z and u_z depends only on the odd powers z and even powers of ρ. Taking into account that the displacements should satisfy boundary conditions (11), we write

$$u_\rho = A_1\rho + A_2\rho^3 + A_3\rho z^2, \qquad u_z = B_1 z + B_2 z^3 + B_3 z\rho^2. \tag{24}$$

The final length of the chosen power series is justified by the exact solution obtained below.

In the examined case, the equilibrium equation $\nabla \cdot T = 0$ has two nontrivial components

$$\frac{\partial T_{\rho\rho}}{\partial \rho} + \frac{\partial T_{\rho z}}{\partial z} + \frac{T_{\rho\rho} - T_{\theta\theta}}{\rho} = 0, \qquad \frac{\partial T_{\rho z}}{\partial \rho} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{\rho z}}{\rho} = 0; \tag{25}$$

and boundary conditions (11) become

$$T_{\rho\rho}\rho + T_{\rho z} z = M^2 z^2 \rho/2, \qquad T_{\rho z}\rho + T_{zz} z = M^2 \rho^2 z/2. \tag{26}$$

Here it is taken into account that the components ρ and z on the surface of a unit sphere are the corresponding projections of the outward normal vector.

Let us substitute series (24) into the relations between the stress and strain tensors (14) and (15) and the obtained results into Eqs. (25) and (26). The equilibrium equations should be satisfied for any ρ and z, and the boundary conditions for any values the latitudinal angle α (for the points on the boundary of the sample, $\rho = \sin\alpha$ and $z = \cos\alpha$). A combination of these conditions leads to a system of six linear equations with a nondegenerate determinant. Solving this system by a standard method, we find the coefficients A_i and B_i of expansions (24). Using the expressions obtained, for the displacement vector components, we obtain

$$u_\rho(\rho, z) = \frac{M^2 \rho}{2(19\beta+14)} \Big(-\frac{2(4\beta+3)\beta}{3\beta+2} + \beta\rho^2 + (3\beta+7)z^2 \Big);$$

$$u_z(\rho, z) = \frac{M^2 z}{2(19\beta+14)} \Big(\frac{16\beta^2 + 31\beta + 14}{3\beta+2} - 2\beta z^2 - (4\beta+7)\rho^2 \Big),$$

whence, using the definition (16), we find the elongation parameter

$$\varepsilon = \frac{u_z(0, R)}{R} = M^2 \, \frac{10\beta^2 + 27\beta + 14}{2(19\beta+14)(3\beta+2)}.$$

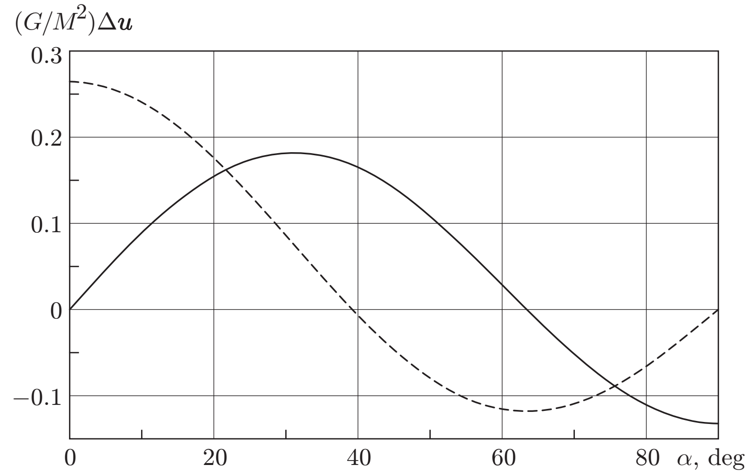

Fig. 1. Differences in displacement on the boundary of a spherical sample between the exact solution and the classical estimate (spheroid) versus the angle α: the solid curve refers to the ρ-component and the dashed curve to the z-component.

For an incompressible material ($\beta \to \infty$), the displacement field is given by

$$u_z = -M^2 z(3z^2 + 6\rho^2 - 8)/57, \qquad u_\rho = M^2 \rho(3\rho^2 + 9z^2 - 8)/114,$$

and the elongation equals

$$\varepsilon = 5\mu_0 M^2/(57G). \tag{27}$$

The last formula is written in dimensional quantities.

Knowing the vector $\boldsymbol{u}$, it is possible to find the total stress and strain fields inside the sample:

$$\begin{gathered} e_{\rho\rho} = (1/2)M^2(-8 + 9\rho^2 + 9z^2)/57, \qquad e_{zz} = -M^2(-8 + 9z^2 + 6\rho^2)/57, \\ e_{\theta\theta} = (1/2)M^2(-8 + 3\rho^2 + 9z^2)/57, \qquad e_{\rho z} = -(1/2)M^2 z\rho/19; \end{gathered} \tag{28}$$

$$\begin{gathered} T_{\rho\rho} = (1/2)M^2(1 - \rho^2 + 20z^2)/19, \qquad T_{zz} = (1/2)M^2(17 - 15\rho^2 + 2z^2)/19, \\ T_{\theta\theta} = (1/2)M^2(1 - 5\rho^2 + 20z^2)/19, \qquad T_{\rho z} = -M^2 z\rho/19. \end{gathered} \tag{29}$$

Formulas (28) and (29) represent the stress and strain components. We recall that for infinitesimal external perturbations, the strain and stress field are constructed in an undeformed sample. As one can see, the largest values of the diagonal tensor stress components are reached at the poles of the sphere. On the equator of the sample, the component $T_{\theta\theta}$ is negative, and in the polar zones, a thin surface layer is formed, in which the material undergoes longitudinal compression ($e_{zz} < 0$). The largest tensile strains are on the central part of the sample, and the maximal shear strains are on the surface belts located at $\alpha = 45°$ and $135°$. Generally, exact calculations show that even in a sphere made of a homogeneous ferroelastic material and even in an indefinitely small, uniform magnetic field, the magnetic-deformation effect generates a nonuniform strain field. This negates the postulates of stress and strain homogeneity and unchanged ellipsoidal shape, which underlie the classical solution [9]. In other words, as shown above, the body to which a sphere is transformed under magnetic deformation is not an ellipsoid of revolution.

A quantitative estimate of the difference between the exact calculation of the MDE and the approach using a spheroid as a variational solutions can be obtained by comparing the coefficients in relations (18) and (27). It is obvious that the error that arises from the choice of a spheroid as a variational solution exceeds 30%. In other words, if one uses a spherical ferrogel sample to measure Young's modulus using the MDE (say, in determining the quantity $\mu_0 M^2/\varepsilon$), data interpretation using the classical estimate yields a value of G that is approximately one-third smaller than the true value.

Figure 1 gives curves of the latitudinal angle α versus the difference between the exact solution and the classical estimate (spheroid) for the displacement vector components on the sample boundary. It is assumed that

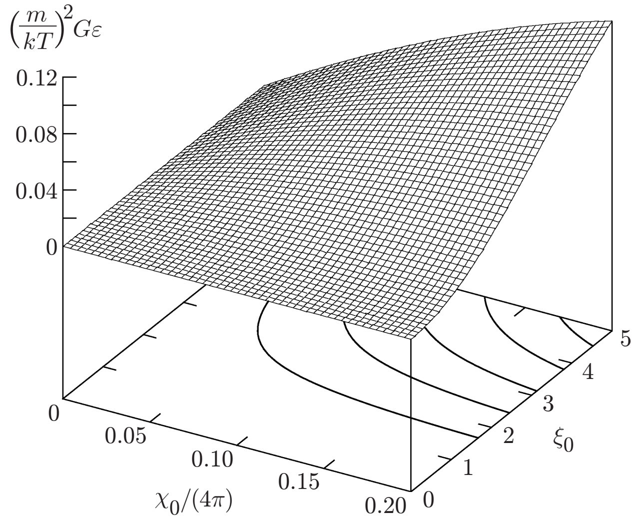

Fig. 2. Elongation parameter of a sphere as a function of the initial susceptibility of a ferroelastic material and the dimensionless strength of the applied field.

the center of the body always remains immovable. It is obvious that the body into which the sphere is transformed as a result of the MDE is longer in the longitudinal direction and hence, narrower in the cross section than an ellipsoid of revolution of equal volume. Qualitatively, this conclusion already follows from the relationship between the coefficients in formulas (27) and (18). We note, however, that the representation of the MDE via magnetization does not allow one to see the direct relationship between the magnetic strain of the sphere and the factors responsible for this. This is due to the complex dependence between the field applied to the magnetized sample and the internal field in it. Indeed, relation (7) and relation (9) following from it are transcendental equations for the internal field, from which magnetization is then determined, for example, according to Eq. (8). The results of such transformation are illustrated in Fig. 2, in which curves of the MDE versus the initial susceptibility of a ferroelastic material χ_0 and the dimensionless strength ξ_0 the applied magnetic field are plotted using exact values. The first of the indicated quantities is a directly measured characteristic of the material, and the second is a parameter which is directly controlled in experiments. We pay attention to the choice of the scale for measuring ε; it is motivated by the necessity of representing the right side of (27) as a function of only χ_0 and ξ_0. Since an increase in any of these quantities leads to an increase in the magnetostatic energy gain during sample elongation, the monotonic increase in ε in Fig. 2 is quite clear.

Numerical Solution of the Linear Problem of the Magnetic-Deformation Effect for an Ellipsoid. We extend the formulation of the problem and consider the situation in which the sample in the initial state has the shape of an arbitrary ellipsoid of revolution. In this case, the magnetic field is still applied along the principal axis of symmetry. Since the exact solution of the elastic problem can be found only for a spherical sample, for a spheroidal initial configuration, we employ a finite-element method. As the basis, we use a complete system of linearly independent functions ϕ_j, which is constructed as follows. The axial section of the ellipsoid of revolution is divided into triangles, and each top of the triangles is taken to be a node. All nodes are numbered from 0 to $m-1$. Each function ϕ_j is considered piecewise-linear; it is equal to unity at the jth node and zero at the remaining nodes.

The scalar functions ϕ_j are used to construct a system of vector functions $\{\boldsymbol{\Phi}_j\}$ that possess the same properties as $\{\phi_j\}$. Bearing in mind that the examined problem possesses axial symmetry, we write the indicated system in cylindrical coordinates:

$$\boldsymbol{\Phi}_{2j} = \phi_j \boldsymbol{\epsilon}_\rho, \qquad \boldsymbol{\Phi}_{2j+1} = \phi_j \boldsymbol{\epsilon}_z, \qquad j = 0, \ldots, m-1.$$

We expand the displacement field inside the sample in terms of the chosen system of functions

$$\boldsymbol{u} = \sum_{j=0}^{2m-1} u_j \boldsymbol{\Phi}_j$$

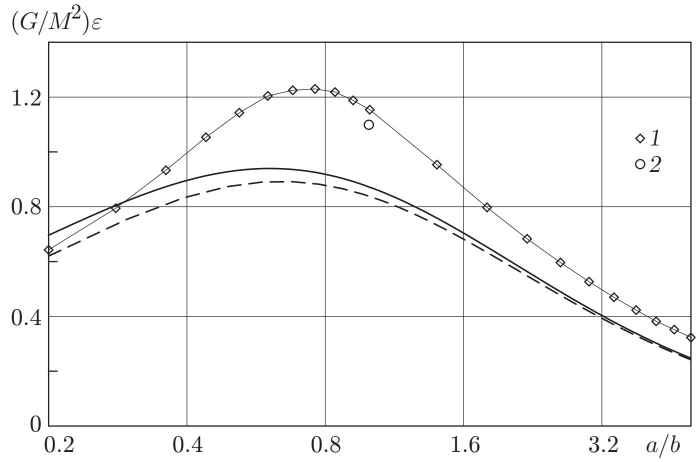

Fig. 3. Elongation of a spheroid versus the initial value of a/b (compressibility parameter $\beta = 20$): the solid curve refers to calculations using formulas (19) and (21) of the classical solutions; the dashed curve refers to the same for an incompressible material ($\beta = \infty$); points 1 refer to the numerical calculation and points 2 refer to the exact solution for a sphere from an incompressible material.

and substitute this expansion into the variational equation (17), again assuming that the elastic state equation is given by Hooke's law (10). Setting the coefficients at δu_j equal to zero, we obtain a system of $2m$ linear algebraic equations for the amplitudes u_k:

$$\sum_{k=0}^{2m-1} u_k \int_V (\nabla \cdot \boldsymbol{\Phi}_k \nabla \cdot \boldsymbol{\Phi}_j + \beta(\nabla \boldsymbol{\Phi}_k + \nabla \boldsymbol{\Phi}_k^{\mathrm{t}}) \cdot \nabla \boldsymbol{\Phi}_j)\, dV = \frac{1}{2} \int_\Gamma \boldsymbol{n} \cdot \boldsymbol{\Phi}_j M_n^2 \, dS. \tag{30}$$

System (30) is solved by standard methods for various values of the material parameters of a ferroelastic material and the aspect ratio a/b of the initial configuration. The resulting displacement field is used to find the stress and strain tensors and then the elongation parameter ε from formula (12). The quantity ε obtained by numerical calculation of the MDE is shown in Fig. 3 by a dashed curve as a function of the ratio a/b of the initial spheroid; here the point $a/b = 1$ corresponds to a spherical sample. The solid curve shows the dependence obtained using formulas (19) and (21). It can be called a generalized classical solution since it is obtained by extension of the MDE problem from [9] (the hypothesis of stress and strain homogeneity) to the case where the initial shape of the body is not a sphere but a spheroid. As can be seen, there is complete qualitative similarity between the curves. Quantitatively, the maximum on the curve of $\varepsilon(a/b)$ that corresponds to the exact solution is more pronounced. A common feature of both approaches is a decrease in the MDE (a decrease in ε) for strongly oblate ($a/b \to 0$) and strongly prolate ($a/b \gg 1$) samples. The latter is an expected effect: for the classical formulation of the problem, this tendency is established by relations (23). In other words, the magnetic stretching of rodlike and flat lenticular spheroids from ferroelastic materials is extremely hindered. As can be seen from Fig. 3, the stretching conditions are optimal, i.e., the magnetic-deformation susceptibility is maximal for slightly oblate bodies: $a/b \simeq 0.8$.

Figure 3 shows the behavior of the numerical coefficient in the formulas generalizing (18) and (27), respectively, for $M = \text{const}$, thus illustrating the geometrical dependence of the elastic effect for a specified internal magnetic state of the sample. To relate the elongation ε to the applied field and the magnetic characteristics of a ferroelastic material, it is necessary, as was done above in constructing Fig. 2, to find a solution $\xi(\xi_0, \chi_0)$ of Eq. (9) and to use it in the magnetic state equation (8). The results of this transformation are given in Figs. 4 and 5. Figure 4 shows how the geometrical effect in the MDE changes with change in the applied field. Of course, the maxima at $\xi_0 = \text{const}$ on the curves of $\varepsilon(a/b)$ have the same nature as the curves presented in Fig. 3. Since the Langevin function is saturated in strong fields, it is obvious that the curves of $\varepsilon(\xi_0)$ possess the same property for fixed initial values of the ratio a/b. Another consequence of the magnetization saturation is the occurrence of the limiting value of the coordinate of the maximal elongation point on the axis a/b. One can see this by considering the relative position of the level lines plotted on the plane $\varepsilon = 0$ (Fig. 4).

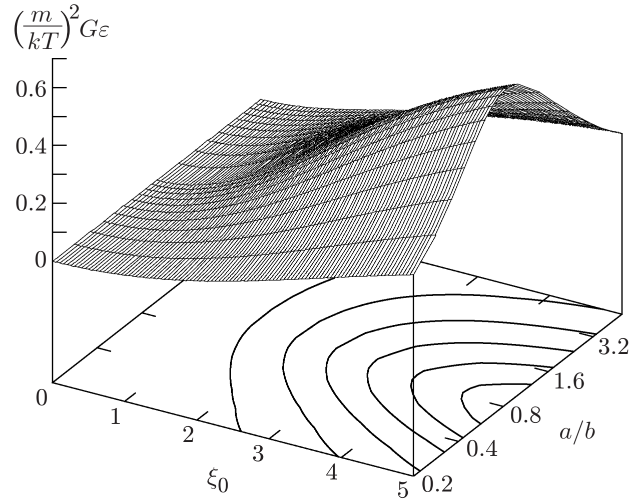

Fig. 4. Elongation of a spheroid as a function of the initial value of a/b and the applied field ξ_0: the initial susceptibility $\chi_0 = 0.1/(4\pi)$; the scale on the axis a/b is logarithmic.

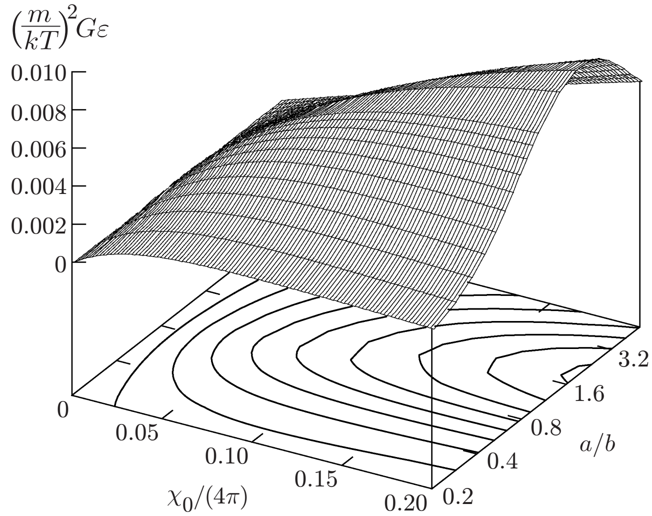

Fig. 5. Elongation of a spheroid as a function of the initial value of a/b and the initial susceptibility $\chi_0/(4\pi)$: the applied field is $\xi_0 = 1$; the scale on the axis a/b is logarithmic.

Figure 5 illustrates the relationship between the geometrical elongation effect and the basic material parameter of a ferroelastic material — its initial magnetic susceptibility $\chi_0 = n\mu_0 m^2/(3kT)$. A distinguishing feature that is easy to find in examining the arrangement of the level lines is the nonmonotonic behavior of the function $\varepsilon(\chi_0)$ for $a/b = \text{const}$. To understand the origin of the indicated line of maxima, we note the following. In a completely nonmagnetic elastomer (magnetic-phase concentration $n = 0$), there is no MDE. However, it occurs in a material that possesses indefinitely weak magnetic susceptibility. Indeed, for the elongation parameter in the scale chosen in Fig. 5, we have $\mu_0(m/kT)^2 G\varepsilon \propto \chi_0^2 L^2(\xi)$. In the case of small χ_0, where the contribution of the susceptibility to the internal field is insignificant, this indicates quadratic growth. The same effect should also occur for rather prolate samples since Eq. (9) contains the susceptibility multiplied by the longitudinal demagnetizing factor. In the case of not too low susceptibilities and not too prolate samples, where the product $3\chi_0 N$ becomes comparable to unity, it is necessary to take into account that the argument of the Langevin function (8) is the internal field ξ defined by Eq. (9). For the case presented in Fig. 5, this equation should be solved for $\xi_0 = \text{const}$. Under these conditions for $\xi_0 \gtrsim 1$, the internal field ξ is a decreasing function of the magnetic susceptibility of the material, resulting in a decrease in the achieved magnetization of the material and, thus, in a decrease in the MDE. In other words, increased magnetic susceptibility leads to shielding of the sample interior from the magnetizing effect of the field applied from outside.

Conclusions. A system of equilibrium equations for a ferroelastic material in the presence of an external magnetic field was proposed. The boundary conditions for the main variables of the problem were formulated, and the simplest constitutive relations were derived. The model medium is a Hooke's elastic continuum magnetized under Langevin's law. It was shown that the magnetic-deformation effect that arises in such a material on exposure to a uniform magnetic field is qualitatively similar to the dielectric striction effect described in [9].

In addition, it was shown that the standard assumption that a sphere or an ellipsoid of revolution subjected to the MDE always keep a spheroidal shape is equivalent to the assumptions that the strain field and the magnetic field in the sample always remain uniform. Direct calculations free from this restriction yielded an analytical solution of the MDE problem for a spherical sample and a numerical solution for any ellipsoid of revolution. It turned out that the difference between the results of exact calculations and the classical approach reaches 30%. Conditions were obtained under which the MDE is maximal for an ellipsoid of revolution of specified volume. It was found that in a certain range of applied fields, an increased content of the magnetic phase in the material (increased initial susceptibility) leads to a decrease rather than an increase in the magnetic-deformation effect.

In summary, we estimate the order of magnitude of the MDE in a soft magnetic material. In a typical case [1, 6], its magnetic phase consists of particles of ferrite, magnetite or gamma-oxy iron of diameter $d \approx 10$ nm and saturation magnetization $I_S \approx 300$ kA/m. The Young's modulus of the matrix (polyvinyl alcohol or polyacrylamide gel) is $G \approx 10^3$ Pa. The concentration of ferroparticles ϕ, according to [6], reaches several volume percent. Assuming that $\phi = 5\%$, we obtain a numerical concentration $n \approx \phi/d^3 \approx 10^{23}$ m^{-3}. Estimation of the value of the magnetic moment of a separate particle (in the chosen range of sizes, it is in a one-domain state) yields $\mu_0 m \approx \mu_0 I_S d^3 \approx 2 \cdot 10^{-25}$ T$\cdot$m^3. From this, the initial susceptibility at room temperature is $\chi_0 = \mu_0 n m^2/(3kT) \approx 0.3$. For the estimated saturation magnetization of the ferrogel, we have $M_0 = nm \approx 10$ kA/m. Estimation of the magnetic-deformation elongation parameter using formula (27) shows that in the order of magnitude, it is $\varepsilon \simeq \mu_0 M_0^2/(3G) \approx 0.1$. In spite the possible inaccuracy of the data used, the last value shows that a magnetic-deformation effect of tens percent is quite achievable.

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